

# Piecewise constant potentials and discrete ambiguities

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## Abstract

This work is devoted to the study of discrete ambiguities. For parametrized potentials, they arise when the parameters are fitted to a finite number of phase-shifts. It generates phase equivalent potentials. Such equivalence was suggested to be due to the modulo  $\pi$  uncertainty inherent to phase determinations. We show that a different class of phase-equivalent potentials exists. To this aim, use is made of piecewise constant potentials, the intervals of which are defined by the zeros of their regular solutions of the Schrödinger equation. We give a classification of the ambiguities in terms of indices which include the difference between exact phase modulo  $\pi$  and the numbering of the wave function zeros.

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## 1 Introduction

Attempts to determine the potential from scattering data have a long history, and conditions to obtain a unique answer are well known (see for instance the textbooks by Newton [1], Chadani and Sabatier [2]). Approaches to the three-dimensional inverse scattering problem can be classified in two categories [1, 2]. In the first case, known as fixed- $\ell$  problem, the potential can be constructed from the phase-shifts  $\delta(\ell, k)$ , if they are known for all momenta  $k \in (0, \infty)$ , and from the discrete spectrum (bound state energies and the corresponding normalization constants). Note that the potential is assumed to satisfy adequate integrability conditions [2]. In the second case, the so-called fixed- $E$  problem, Loeffel [3] has obtained theorems ensuring a unique potential from the knowledge of the phase-shifts  $\delta(\ell, k)$  for all (non-discrete) non negative values of  $\lambda = \ell + 1/2$ . If the data set reduces to discrete values of  $\lambda = \ell + 1/2$  for non-negative integer  $\ell$ , the Carlson's theorem [4] predicts a unique potential  $V(r)$ , provided it belongs to a suitable class [2, 3].

The present work is dealing with the second aspect of the problem. It is relevant to the case where a chosen parametric form is used to fit a differential cross-section at a fixed energy. Here, the uniqueness is defined in the sense of a best fit to a differential cross section, for instance. The argument does not apply to phase shifts, which can be extracted from the scattering amplitude, since a phase is defined *mod*  $n\pi$ . Such undeterminations are called phase- or discrete ambiguities. They have been noticed long ago by Drisko, Satchler and Bassel [5].

As a concrete example, let us quote the discrete ambiguities described by Goldberg *et al* [6]. By using potentials of Woods-Saxon shape, these authors have extracted four  $\alpha - {}^{208}\text{Pb}$  potentials, characterized by depth differences of 50 MeV in the real potential part, giving similar  $\chi$ -square per degree of freedom. The  $\chi$ 's square are related to the phase-shifts via the fit to the differential cross sections. The Jeffreys, Wentzel, Kramers and Brillouin (JWKB) phase-shifts have been calculated for the real parts of the four potentials. They were found to differ by a factor  $n\pi$ ,  $n$  integer and independent of  $\ell$  for low enough angular momentum ( $\ell \leq 44$ ).

Attempts to study these ambiguities are scarce. Sabatier [7] and Cuer [8] have studied their origin by means of the JWKB approximation. A physical interpretation has been given by Leeb and Schmid [9], in which the occurrence of discrete ambiguities is linked to the existence of partly Pauli forbidden states.

Note that the discrete ambiguities are still of interest. Let us mention the recent papers on the subject by Brandan [10] and by Amos and Bennett [11]. These works have been done in the context of heavy and light nuclear ion scattering, respectively.

The purpose of the present work is a further study of the discrete ambiguities. In order to discuss this problem in detail, and stress the origin of the modulo  $\pi$  shift, we shall consider piecewise-constant real potentials. Such potentials have been used recently by Ramm and Gutman [12]. These authors have shown that different piecewise-constant positive real potentials can lead to almost the same fixed-energy phase-shifts. As we shall see, this is a convenient starting point for the present study. Here we enlarge the study and consider also negative potentials. Moreover, we found it particularly useful to restrict the class of potentials to the subset of piecewise-constant potentials the intervals of which are defined by the zeros of the regular solution of the Schrödinger equation.

The present work shows that piecewise constant potentials are a basis for a class of phase equivalent potentials, with a phase ambiguity of  $n\pi$ . Within each  $n$ , these potentials can be ordered according to the positions of the zeros of the regular solutions. The value of  $n$  has a minimum, which can be calculated by means of the JWKB approximation.

Dealing with the exact solution of the Schrödinger equation, our work goes beyond, for a certain class of potentials, the earlier attempts by Sabatier [7] and Cuer [8], who have resorted to the JWKB approximation. The drawback is that our method becomes very tedious as the number of partial waves increases.

In what concerns numerical applications, the energy will be put equal to unity, without loss of generality. Indeed, use is made of the following scaling property: if  $\delta(\ell, k=1)$  is the phase for the potential  $V(r)$  at the energy  $E=1$ , it is also the phase for the potential  $k^2V(kr)$  at the energy  $E=k^2$ .

The paper is organized as follows. In section 2 the basic formalism is recalled. In section 3 our results are presented. The results are discussed in the JWKB approximation in section 4 and our conclusions are presented in section 5.

## 2 Formalism

In this paper we investigate to what extent piecewise constant potentials allow us to reproduce a finite set of fixed-energy phase-shifts. Considering spherically symmetric potentials of finite range, we start from the reduced radial Schrödinger equation. For a positive energy  $E=k^2$  and an angular

momentum  $\ell$  ( $\hbar = 2m = 1$ ), it reads

$$\psi_\ell(k, r)'' + \left( k^2 - v(r) - \frac{\ell(\ell+1)}{r^2} \right) \psi_\ell(k, r) = 0. \quad (1)$$

Here, the reduced potential is defined by

$$v(r) = \frac{2m}{\hbar^2} V(r), \quad (2)$$

in terms of the potential  $V(r)$ , and  $\psi_\ell(k, r)$  is the regular solution of Eq.(1) defined by the Cauchy condition  $\lim_{r \rightarrow 0} \psi_\ell(k, r) r^{-\ell-1} = 1$ .

Our construction of phase-equivalent potentials is based upon the following property. Suppose the function  $\psi_\ell(k, r)$  to vanish for  $r = r(\ell, k)$ . It is unnecessary to know the potential for  $r < r(\ell, k)$  to determine the phase shift  $\delta(\ell, k)$ . This holds for  $(\ell, k)$  fixed. Thus, it is sufficient to consider the Schrödinger equation on the interval  $[r(\ell, k), \infty[$ . Actually, the derivative  $\psi'_\ell$  with respect to  $r$  is undetermined at  $r = r(\ell, k)$ , which has no consequence since the phase-shift is determined only by the ratio  $\psi_\ell/\psi'_\ell$ . Furthermore, for a fixed energy and a fixed  $\ell$ , there is a countable number of zeros [13]. The functions  $\psi_\ell(k, r)$  and  $\psi'_\ell(k, r)$  cannot vanish at the same value of  $r$  for  $2\ell + 1 > 0$  (except for  $r = 0$ ) and  $\ell > 0$  as shown in [14]. Consequently,  $\psi_\ell(k, r)$  has only simple zeros (except at  $r = 0$ ). We recall that thanks to the fact that the zeros are simple, they satisfy the strict ordering  $0 < \hat{r}_1(\ell, k) < \hat{r}_2(\ell, k) < \dots < \hat{r}_n(\ell, k) < \dots < R$ .

The  $n$ th zeros of the regular solution satisfy a monotony property, namely, for fixed  $n$ , the function

$$\ell \mapsto \hat{r}_n(\ell, k) \quad (3)$$

increases with  $\ell$  as has shown by Sturm in the 1830s [15]. More precisely we have

$$\frac{d}{d\ell} \hat{r}_n(\ell, k) = (2\ell + 1) \frac{\int_0^{\hat{r}_n(\ell, k)} dr' \psi^2(k, r')}{\left( \frac{\partial}{\partial r} \psi_\ell(k, \hat{r}_n(\ell, k)) \right)^2}, \quad (4)$$

which is positive definite. As stated before,  $\psi_\ell$  and  $\psi'_\ell$  cannot vanish simultaneously for  $2\ell + 1 > 0$ , except for  $r = 0$  and  $\ell > 0$  [14], so that the denominator is never equal to zero.

To find potentials reproducing  $N + 1$  fixed energy  $E = k^2$  phase-shifts,  $\delta(\ell_j, k)$ ,  $\ell = 0, 1, \dots, N$ , we apply the following procedure :

1. We construct a piecewise constant potential with a compact support:  $v(r) = 0$  for  $r \geq R$ ,  $R$  being fixed.
2.  $\hat{r}_n(\ell, k)$  is the  $n$ -th zero of the regular solution  $\psi_\ell(k, r)$  of the Schrödinger equation (1) for the  $\ell$ -wave aside from the trivial zero at the origin. We denote by  $r_\ell$  a generic non trivial zero of  $\psi_\ell(k, r)$ . For the zero potential  $\hat{r}_n(\ell, k)$  is given in terms of the zeros of the spherical Bessel function  $j_\ell(z)$  labelled  $x_{\ell, n}$ . We have  $k\hat{r}_n(\ell, k) \equiv x_{\ell, n}$ .
3. The potential  $v$  of Eq.(2) is assumed to be constant on each interval delimited by the partition  $0 < r_1 < r_2 < r_3, \dots < r_N < R$ . We set

$$\begin{aligned} v(r) &= v_0, \quad K_0^2 = k^2 - v_0 \quad ; \quad 0 \leq r < r_1 \\ v(r) &= v_1, \quad K_1^2 = k^2 - v_1 \quad ; \quad r_1 \leq r < r_2 \\ &\dots \\ v(r) &= v_N, \quad K_N^2 = k^2 - v_N \quad ; \quad r_N \leq r < R \end{aligned} \quad (5)$$

4. The regular  $u_\ell(kr)$  and irregular  $w_\ell(kr)$  solutions of Eq. (1) for  $v \equiv 0$  are denoted, respectively,

$$\begin{aligned} u_\ell(x) &= \sqrt{\frac{\pi x}{2}} J_{\ell+1/2}(x) \\ w_\ell(x) &= -\sqrt{\frac{\pi x}{2}} Y_{\ell+1/2}(x) \end{aligned}$$

in terms of the Bessel functions  $J_\nu, Y_\nu$  of order  $\nu$ , given in [16]. We have  $u_\ell(x) = x j_\ell(x)$  where  $j_\ell$  is the spherical Bessel function of order  $\ell$ .

5. Let  $\psi_\ell$  be the regular solution of the Schrödinger equation (1). We denote by  $A(r), B(r), C(r), D(r), \dots$  the value of the ratio

$$\frac{\psi_\ell(k, r)}{\psi'_\ell(k, r)} \quad (6)$$

for  $\ell = 0, 1, 2, 3, \dots$  respectively. Here, the prime denotes the first derivative with respect to  $r$ .

As we shall see, these requirements give rise to a class of phase-equivalent potentials.

To show phase-ambiguities, it is worth to recall that the exact phase can be calculated by using the variable phase method of Calogero [17]. Most of the time it allows the removal of the  $n\pi$  ambiguity burdening the direct solution. With this method, the phase-shift is reached by solving a first order differential equation

$$\frac{\partial}{\partial r} \delta(\ell, k, r) = -\frac{v(r)}{k} (u_\ell(kr) \cos(\delta(\ell, k, r)) + w_\ell(kr) \sin(\delta(\ell, k, r)))^2, \quad (7)$$

with  $\delta(\ell, k, 0) = 0$  as boundary condition. The phase-shift is given by the limit  $\delta(\ell, k) = \lim_{r \rightarrow \infty} \delta(\ell, k, r)$ . In the present work, use is made of the variable phase method to determine the exact phases. In other words,  $k$  being fixed, we solve Eq.(7) for each value of  $r$  starting from  $r = 0$ , combined with  $\delta(\ell, k, 0) = 0$ . Actually,  $\lim_{r \rightarrow \infty} \delta(\ell, k, r) = \delta(\ell, k, R) \equiv \delta(\ell, k)$ .

A variant of the Calogero's equation but for  $\tan(\delta(\ell, k, r))$  has been derived in [18]. It can be obtained from Eq.(7) noticing that

$$\frac{\partial}{\partial r} \tan(\delta(\ell, k, r)) = \frac{1}{\cos(\delta(\ell, k, r))^2} \frac{\partial}{\partial r} \delta(\ell, k, r). \quad (8)$$

Setting  $T(\ell, k, r) = \tan(\delta(\ell, k, r))$ , it yields

$$\frac{\partial}{\partial r} T(\ell, k, r) = -\frac{v(r)}{k} (u_\ell(kr) + w_\ell(kr)T(\delta(\ell, k, r)))^2, \quad (9)$$

which is solved with the boundary condition  $T(\ell, k, 0) = 0$ . At zero energy, the latter formalism allows the calculation of both the scattering length and the effective range directly from the potential. In our case, we have chosen to work with Eq.(7) to avoid the numerical difficulties owing to poles in  $T(\ell, k, r)$ .

### 3 Examples

#### 3.1 A single phase-shift

Although the case of a single known phase-shift is trivial, it illustrates features of the discrete ambiguities.

Assume, for example, the s-wave phase-shift  $\delta(\ell = 0, k) \equiv \delta_0$  to be known. The procedure amounts to find a constant  $v_0$  in  $[0, R]$ . Setting

$$\frac{\tan(kR + \delta_0)}{k} = A(R) , \quad (10)$$

the equation to be solved is, depending upon whether  $K_0^2$  is positive or negative

$$\begin{aligned} \frac{\tan(K_0 R)}{K_0} &= A(R) , & K_0 &= \sqrt{K_0^2} \\ \frac{\tanh(K_0^* R)}{K_0^*} &= A(R) , & K_0^* &= \sqrt{-K_0^2} . \end{aligned} \quad (11)$$

For  $K_0^2 \geq 0$ , discrete ambiguities come from the fact there is a countably infinite set of solutions to Eq.(11). Whatever the right-hand-side of (11), one solution of (11) exists in the interval

$$(2n-1)\frac{\pi}{2} \leq \sqrt{k^2 - v_0} R < (2n+1)\frac{\pi}{2} \quad (12)$$

for every  $n \geq 1$ . In  $[0, \pi/2[$  the solution exists only for suitable values of the r.h.s. of the equation. The difference between the solutions for  $v_0$  on two adjacent intervals is of the order of the square of the length of the interval, namely  $\pi^2$ . At the potential level, it introduces a difference of  $\Delta V \sim \frac{\pi^2 \hbar^2}{2mR^2}$ . As a consequence, the larger is the mass  $m$ , the smaller are the differences between phase equivalent potentials, in agreement what is observed phenomenologically [19]. It means that while searching optical potential parameters, the chance to find phase equivalent potentials on a finite parameter space is larger for heavy than for light systems. Indeed discrete ambiguities are often observed for nucleus-nucleus scattering with a heavy target, for instance for  $\alpha - {}^{208}\text{Pb}$  scattering (not so for  $\alpha - {}^{58}\text{Ni}$ ), whereas few attempts have been made for nucleon-nucleus scattering. Note that for proton scattering on Cr isotopes at 10 MeV, discrete ambiguities have been reported by Andrews *et al.* [20]. The difference in the potential depth is about 100 per cent. At high energies  $|v_0|$  is weak as compared to  $k^2$ , and we expect  $\sqrt{k^2 - v_0}R$  to fall in the same interval as  $kR$ , namely

$$(2n_k-1)\frac{\pi}{2} \leq k R < (2n_k+1)\frac{\pi}{2} . \quad (13)$$

This is due to the fact that  $\sqrt{k^2 - v_0}R \sim kR - v_0 R/(2k)$ . As a consequence, the solution cannot be found in another interval, and the discrete ambiguities disappear. This is confirmed by the analysis of scattering data at high energies. Finally, to finish with this example, we have verified that the different solutions of Eq. (12) lead to s-wave phase shifts which differ by multiples of  $\pi$ .

### 3.2 2 phase-shifts $\ell = 0, 1$

The determination of the piecewise constant potential starts from the largest zero  $r_N < R$  fitting the solution  $v_N(K_N)$  on the largest  $\ell = N$  phase-shift considered. Finding  $v_{N-1}(K_{N-1})$  on the previous interval  $[r_{N-1}, r_N[$  is more complicated in general, as it depends not only on  $r_{N-1}$  but also from  $r_N, K_N$ . The complexity increases with the number of zeros considered. Two exceptions exist, namely the  $\ell = 0, 1$  and  $\ell = 0, 1, 2$  cases. These exceptions are called simply soluble, and their solutions are presented below.

The regular solution for  $\ell = 1$  is denoted  $u_1(K_0 r)$  for  $r < r_1$ . Since  $r_1$  is a zero of  $\psi_1$ , we have  $K_0 r_1 = x_{1,n_1}$ , and

$$(\forall n_1) \ u_1(x_{1,n_1}) = \sin(x_{1,n_1})/x_{1,n_1} - \cos(x_{1,n_1}) \equiv 0 . \quad (14)$$

On the interval  $[r_1, R[$ , up to a constant factor, it reads :

$$\psi_1(k, r) = u_1(K_1 r) w_1(K_1 r_1) - u_1(K_1 r_1) w_1(K_1 r) . \quad (15)$$

Finally, for  $r \geq R$  the regular solution is proportional to

$$\psi_1(k, r) = u_1(kr) \cos(\delta_1) + w_1(kr) \sin(\delta_1) . \quad (16)$$

Equating the inverse of the logarithmic derivative of solutions (15,16) at  $r = R$ , we obtain

$$\begin{aligned} \frac{\tan(K_1(R - r_1))}{K_1} &= \frac{\mathcal{N}}{\mathcal{D}} & K_1 &= \sqrt{K_1^2} \\ \frac{\tanh(K_1^*(R - r_1))}{K_1^*} &= \frac{\mathcal{N}}{\mathcal{D}} & K_1^* &= \sqrt{-K_1^2} . \end{aligned} \quad (17)$$

Here we have

$$\begin{aligned} \mathcal{N} &= -B(R)R - R^2 + B(R)r_1 + Rr_1 - B(R)K_1^2 R^2 r_1 \\ \mathcal{D} &= -B(R) - R + B(R)K_1^2 R^2 - B(R)K_1^2 Rr_1 - K_1^2 R^2 r_1 . \end{aligned} \quad (18)$$

We recall that in these expressions,  $B(R) = (\psi_1/\psi_1')(k, R)$ , with  $\psi_1$  given by Eq.(16), at  $r \geq R$ .

Turning to the  $\ell = 0$  wave, the regular solution in  $[0, r_1[$  is proportional to  $\sin(K_0 r)$ . Therefore,  $(\psi_0/\psi_0')(r = r_1) = \tan(K_0 r_1)/K_0$ , which is equal to  $r_1$  when Eq. (14) is taken into account. At the other end of the interval,  $[r_1, R[$ , the function  $\psi_0/\psi_0'$  is given by

$$\frac{\psi_0(k, R)}{\psi_0'(k, R)} = \frac{K_1 r_1 \cos(K_1(R - r_1)) + \sin(K_1(R - r_1))}{K_1(\cos(K_1(R - r_1)) - K_1 r_1 \sin(K_1(R - r_1)))} . \quad (19)$$

It is identified with  $A(R) = \tan(kR + \delta_0)/k$ . We then obtain

$$\begin{aligned} \frac{\tan(K_1(R - r_1))}{K_1} &= \frac{A(R) - r_1}{1 + A(R) K_1^2 r_1} & K_1 &= \sqrt{K_1^2} \\ \frac{\tanh(K_1^*(R - r_1))}{K_1^*} &= \frac{A(R) - r_1}{1 + A(R) K_1^2 r_1} & K_1^* &= \sqrt{-K_1^2} . \end{aligned} \quad (20)$$

Equating equations (17), (18) and (20), we are left with

$$K_1^2 = \frac{A(R)B(R) + A(R)R - B(R)R - R^2}{A(R)B(R)R^2} . \quad (21)$$

Provided that  $A(R)B(R) \neq 0$  or equivalently that  $R$  is a zero of neither  $\psi_0$  nor  $\psi_1$ . The equation (21) also reads

$$\begin{aligned} K_1^2 &= k^2 + \frac{\mathcal{N}}{\mathcal{D}} \\ \mathcal{N} &= k^3 R \sin(\delta_0 - \delta_1) \\ \mathcal{D} &= \sin(kR + \delta_0) (\sin(kR + \delta_1) - kR \cos(kR + \delta_1)) . \end{aligned} \quad (22)$$

It fixes the value of  $K_1^2$ , and consequently the constant  $v_1$ , on the interval  $[r_1, R[$  independently of the value of  $r_1$ . For this very reason, the model is quoted as simply soluble. With this value of  $K_1^2$ , Eq. (20) allows us to determine  $r_1$ . In this case, a first source of ambiguities come from the choice of the value of  $r_1$  among the zeros of  $\psi_1$  smaller than  $R$ . The next step consists in determining  $v_0$ . To this aim, use is made of the fact that  $K_0 r_1 = x_{1, n_1}$  is a zero of  $\psi_1 \forall n_1$ . The choice of which zero  $x_{1, n_1}$  of the spherical Bessel function  $j_1$  is another source of ambiguity.

$x_{1,1} = 4.493409$		$x_{1,2} = 7.725252$		
$n$	$K_0$	$n$	$K_0$	$r_1$
-7	0.467351105	-6	0.803489067	9.61463257
-6	0.524561926	-5	0.901848241	8.5660223
-5	0.597764867	-4	1.02770172	7.51701782
-4	0.694775396	-3	1.19448618	6.467427
-3	0.829515446	-2	1.42613676	5.41690836
-2	1.02946743	-1	1.76990236	4.36479003
-1	1.35771439	0	2.33423792	3.30953931
0	1.99999964	1	3.4384809	2.24670498
1	3.89058288	2	6.68884865	1.15494498

Table 1: For the first two zeros of  $j_1(x)$ , the value of  $K_0$  is listed as function of the shift ( $n\pi$ ) with respect to the exact phase of the reference potential  $K_0 = 2, r_1 = 2.24670498, K_1 = 3, R = 10$ .

### Numerical application

To fix ideas, the following example is treated. A reference potential is defined by setting  $v_0 = -3, r \in [0, r_1[, v_1 = -8, r \in [r_1, R[$ . The energy is fixed at  $E = k^2 = 1$ . Thus, we have  $K_0 = 2$  and  $K_1 = 3$ . The first zero of the spherical Bessel function  $j_1$  is chosen, namely  $x_{1,1} = 4.493409$ . It fixes the value of  $r_1$  at 2.2467045. The two phase-shifts  $\delta_0$  and  $\delta_1$  are then calculated by the variable phase method.

With these pseudo-data, we look for phase-equivalent potentials of the same range  $R = 10$ , i.e. potentials giving the same  $\ell = 0, 1$ -phase-shifts modulo  $\pi$ . In fact the phase-equivalent potentials are determined from the values of  $A(R), B(R), R = 10$  calculated from the reference potential. We recall that  $K_1^2$  is recovered from Eq.(21), and is therefore identical to the starting value. In the Table 1 are shown the different values of  $r_1 < R = 10$  obtained from Eqs.(20). Only the first two zeros of the spherical Bessel function  $j_1(z) = u_1(z)/z$  namely  $x_{1,1} = 4.493409, x_{1,2} = 7.725252$ , are displayed in Table 1. They concern the ambiguities relating to  $K_0$ .

The ambiguities proceed from two sources: the different values of  $r_1$  and the choice of the zero  $x_{1,n_1}$ . As a consequence, two different potentials may give the same shift with respect to the exact phase. At this stage, the ambiguities can be characterized by two numbers:

- 1)  $n_1$  : numbering the zero of the regular solution for the p-wave ( $\ell = 1$ )
- 2)  $n$  : the difference between the exact phases of the phase-equivalent and the starting potentials divided by  $\pi$ . In the present example this number happens to be the same for both  $\ell = 0$  and  $\ell = 1$  phases.

The values of  $K_0$  are listed and ordered as function of the  $n\pi$  shift with respect to the exact solution. For  $n = 0$  and  $x_{1,1} = 4.493409$ , the parameters of the reference potential are recovered with a  $10^{-7}$  accuracy. Due to the finite range  $R$ , Table 1 exhausts the number of phase equivalent potentials for the first two zeros of  $j_1$ . Other cases exist, corresponding to the higher zeros of  $j_1$ . Here  $n_1 = 1$  for all  $K_0$  of the second column and  $n_1 = 2$  for the  $K_0$ 's of column 4.

It is interesting to quote results for transparent potentials. They are characterized by  $\delta_0 = \delta_1 = 0$ . It implies automatically  $K_1 = k = \sqrt{E} = 1$  and  $r_1$  takes only the two values 7.72525183 or 4.49340947 smaller than the range parameter  $R = 10$ . A freedom still exists in the choice of  $x_{1,n_1}$  which determines  $K_0$ . The results are displayed in Table 2.

$n$	$n_1$	$K_0$	$r_1$
-1	1	0.581652127	7.72525183
0	1	0.999999931	4.49340947
0	2	1.00000004	7.72525183
1	2	1.71924064	4.49340947
1	3	1.41149086	7.72525183
2	3	2.42669234	4.49340947
2	4	1.82080713	7.72525183
3	4	3.13040547	4.49340947
3	5	2.22915123	7.72525183
4	5	3.83244722	4.49340947
4	6	2.63697599	7.72525183
5	6	4.53359608	4.49340947
5	7	3.0444899	7.72525183
6	7	5.23421053	4.49340947

Table 2: Transparent potentials  $R = 10$ . For two values of  $r_1 = x_{1,1}, x_{1,2}$ ,  $K_0$  is listed as a function of  $n$  defined in the text, together with  $n_1$ , the order of the zero of  $j_1(x)$ .

### 3.3 3 phase-shifts $\ell = 0, 1, 2$

Consider the case where three phase-shifts  $\delta_0, \delta_1, \delta_2$ , corresponding to  $\ell = 0, 1$  and  $2$ , are known. On the interval  $[r_2, R[$ , the  $\ell = 2$  regular solution reads, up to a constant multiplicative factor,

$$\psi_2(k, r) = u_2(K_2 r) w_2(K_2 r_2) - u_2(K_2 r_2) w_2(K_2 r) . \quad (23)$$

It vanishes at  $r = r_2$  as imposed. For  $r \geq R$ , it is proportional to

$$\psi_2(k, r) = u_2(kr) \cos(\delta_2) + w_2(kr) \sin(\delta_2) . \quad (24)$$

With  $C(R) = (\psi_2/\psi'_2)(k, R)$ , the continuity conditions at  $r = R$  require

$$\begin{aligned} \frac{\tan(K_2(R - r_2))}{K_2} &= \frac{\mathcal{N}}{\mathcal{D}} , & K_2 &= \sqrt{K_2^2} \\ \frac{\tanh(K_2^*(R - r_2))}{K_2^*} &= \frac{\mathcal{N}}{\mathcal{D}} , & K_2^* &= \sqrt{-K_2^2} . \end{aligned} \quad (25)$$

Here we have

$$\begin{aligned} \mathcal{N} &= -18C(R)R - 9R^2 + 3C(R)K_2^2 R^3 + 18C(R)r_2 + 9Rr_2 - 9C(R)K_2^2 R^2 r_2 \\ &\quad - 3K_2^2 R^3 r_2 + 6C(R)K_2^2 Rr_2^2 + 3K_2^2 R^2 r_2^2 - C(R)K_2^4 R^3 r_2^2 \\ \mathcal{D} &= -18C(R) - 9R + 9C(R)K_2^2 R^2 + 3K_2^2 R^3 - 18C(R)K_2^2 Rr_2 - 9K_2^2 R^2 r_2 \\ &\quad + 3C(R)K_2^4 R^3 r_2 + 6C(R)K_2^2 r_2^2 + 3K_2^2 Rr_2^2 - 3C(R)K_2^4 R^2 r_2^2 \\ &\quad - K_2^4 R^3 r_2^2 . \end{aligned} \quad (26)$$

Other constraints come from the fact that  $r_2$  is a zero of  $\psi_2$ . First, for  $0 \leq r < r_1$ , the function  $\psi_2$  is proportional to  $u_2$ . At  $r = r_1$ , we have  $(u_2/u'_2)(K_0 r_1) = -r_1/2$ , since for  $x = K_0 r_1$  we have  $\sin(x)/x - \cos(x) = u_1(x) = 0$ . Expanding  $\psi_2$  as a linear combination of  $u_2(K_1 r)$  and  $w_2(K_1 r)$  (in the case where  $K_1^2 \geq 0, K_1 = \sqrt{K_1^2}$ ) on  $[r_1, r_2[$ , taking into account that  $(\psi_2/\psi'_2)(k, r) = -r_1/2$  at



$r = r_1$  and zero at  $r = r_2$ , we are left with:

$$\begin{aligned}\frac{\tan(K_1(r_2 - r_1))}{K_1} &= \frac{\mathcal{N}}{\mathcal{D}} & K_1 &= \sqrt{K_1^2} \\ \frac{\tanh(K_1^*(r_2 - r_1))}{K_1^*} &= \frac{\mathcal{N}}{\mathcal{D}} & K_1^* &= \sqrt{-K_1^2} \\ \frac{\mathcal{N}}{\mathcal{D}} &= \frac{-3r_1 + 3r_2 + K_1^2 r_1 r_2^2}{3 + 3K_1^2 r_1 r_2 - K_1^2 r_2^2}.\end{aligned}\quad (27)$$

For  $\ell = 1$ , the regular solution is zero at  $r = r_1$ , and reads at  $r = r_2$ , up to a non zero multiplicative factor, (when  $K_1^2 \geq 0, K_1 = \sqrt{K_1^2}$ )

$$\psi_1(k, r_2) = u_1(K_1 r_2) w_1(K_1 r_1) - w_1(K_1 r_2) u_1(K_1 r_1), \quad (28)$$

and its derivative is

$$\psi_1'(k, r_2) = K_1(u_1'(K_1 r_2) w_1(K_1 r_1) - w_1'(K_1 r_2) u_1(K_1 r_1)). \quad (29)$$

It implies that

$$\frac{r_2 (r_1 + \mathcal{T} + r_2 (-1 + K_1^2 r_1 \mathcal{T}))}{-r_1 + r_2 - \mathcal{T} - K_1^2 r_1 r_2 \mathcal{T} + K_2^2 r_2^2 (r_1 + \mathcal{T})} = \frac{\psi_1(k, r_2)}{\psi_1'(k, r_2)}. \quad (30)$$

Here we have defined

$$\begin{aligned}\mathcal{T} &= \frac{\tan(K_1(r_2 - r_1))}{K_1} & K_1 &= \sqrt{K_1^2} \\ \mathcal{T} &= \frac{\tanh(K_1^*(r_2 - r_1))}{K_1^*} & K_1^* &= \sqrt{-K_1^2}.\end{aligned}\quad (31)$$

Equating  $\mathcal{T}$  with  $\mathcal{N}/\mathcal{D}$  of Eq.(27) and using Eq.(30), we obtain  $(\psi_1/\psi_1')(k, r_2) = B(r_2) = r_2/2$ .

The  $\ell = 1$  regular solution  $\psi_1$  on  $[r_2, R]$  is a linear combination of  $u_1(K_2 r)$  and  $w_1(K_2 r)$  (when  $K_2^2 \geq 0, K_1 = \sqrt{K_1^2}$ ). Using the constraint that  $(\psi_1/\psi_1')(k, r)$  is equal to  $r_2/2$  at  $r = r_2$  and to  $B(R)$  at  $r = R$  we obtain:

$$\begin{aligned}\frac{\tan(K_2(R - r_2))}{K_2} &= \frac{\mathcal{N}}{\mathcal{D}} & K_2 &= \sqrt{K_2^2} \\ \frac{\tanh(K_2^*(R - r_2))}{K_2^*} &= \frac{\mathcal{N}}{\mathcal{D}} & K_2^* &= \sqrt{-K_2^2} \\ \mathcal{N} &= 3B(R)R + 3R^2 - 3B(R)r_1 - 3Rr_2 \\ &+ 3B(R)K_2^2 R^2 r_2 - B(R)K_2^2 R r_2^2 - K_2^2 R^2 r_2^2 \\ \mathcal{D} &= 3B(R) + 3R - 3B(R)K_2^2 R^2 + 3B(R)K_2^2 R r_2 \\ &+ 3K_2^2 R^2 r_2 - B(R)K_2^2 r_2^2 - K_2^2 R r_2^2 \\ &+ B(R)K_2^4 R^2 r_2^2.\end{aligned}\quad (32)$$

Identifying (25, 26) and (32,33) we have finally

$$K_2^2 = \frac{4B(R)C(R) + 2B(R)R - 2C(R)R - R^2}{B(R)C(R)R^2} \quad (34)$$

or, equivalently, provided that  $R$  is not a zero of  $\psi_1$  or  $\psi_2$

$$\begin{aligned} K_2^2 &= k^2 + \frac{\mathcal{N}}{\mathcal{D}} \\ \mathcal{N} &= k^5 R^3 \sin(\delta_1 - \delta_2) \\ \mathcal{D} &= (\sin(kR + \delta_1) - kR \cos(kR + \delta_1)) \\ &\quad \times ((3 - k^2 R^2) \sin(kR + \delta_2) - 3kR \cos(kR + \delta_2)) . \end{aligned} \quad (35)$$

Here again the value of  $K_2^2$  and of  $v_2$  on  $[r_2, R[$  does not depend on  $r_2$ , and the model is still simply soluble. Once  $K_2^2$  is known, we have to determine  $r_2$  through Eqs. (32,33). Then Eq. (21) expresses  $K_1^2$  in terms of the ratio  $(\psi_\ell/\psi'_\ell)(k, r_2)$  for  $\ell = 0, 1$ , namely the quantities  $A(r_2), B(r_2)$ . We recall that  $(\psi_1/\psi'_1)(k, r_2) = r_2/2 = B(r_2)$ , as shown before. By definition  $A(r_2) = (\psi_0/\psi'_0)(k, r_2)$ . Thus, on the interval  $[r_2, R[$ , the matricial equation for  $K_2^2 \geq 0$

$$\begin{pmatrix} \cos(K_2(R - r_2)) & \frac{\sin(K_2(R - r_2))}{K_2} \\ -K_2 \sin(K_2(R - r_2)) & \cos(K_2(R - r_2)) \end{pmatrix} \begin{pmatrix} \psi_0(k, r_2) \\ \psi'_0(k, r_2) \end{pmatrix} = \begin{pmatrix} \psi_0(k, R) \\ \psi'_0(k, R) \end{pmatrix} \quad (36)$$

allows the determination of  $A(r_2)$  (for  $K_2^2 \leq 0$  the extension of the matricial equation is straightforward). Indeed, the solution of this matricial equation yields

$$A(r_2) = \frac{A(R) - \mathcal{T}}{A(R)K_2^2 \mathcal{T} + 1} , \quad (37)$$

with

$$\begin{aligned} \mathcal{T} &= \frac{\tan(K_2(R - r_2))}{K_2} & K_2 &= \sqrt{K_2^2} \\ &= \frac{\tanh(K_2^*(R - r_2))}{K_2^*} & K_2^* &= \sqrt{-K_2^2} . \end{aligned} \quad (38)$$

To determine  $K_1^2$  on the interval  $[r_1, r_2[$  use is made of Eq. (21) by setting  $R = r_2$ , namely

$$K_1^2 = \frac{A(r_2)B(r_2) + A(r_2)r_2 - B(r_2)r_2 - r_2^2}{A(r_2)B(r_2)r_2^2} , \quad (39)$$

where  $B(r_2) = r_2/2$ .

Let us define

$$F_0 = 54A(R)C(R) + 27A(R)R - 54C(R)R - 27R^2 \quad (40)$$

$$\begin{aligned} F_1 &= -27A(R)C(R)R^2 - 9A(R)R^3 + 9C(R)R^3 + 36A(R)C(R)r_2^2 \\ &\quad + 18A(R)Rr_2^2 - 36C(R)Rr_2^2 - 18R^2r_2^2 + 18C(R)r_2^3 + 9Rr_2^3 \end{aligned} \quad (41)$$

$$\begin{aligned} F_2 &= -18A(R)C(R)R^2r_2^2 - 6A(R)R^3r_2^2 + 6C(R)R^3r_2^2 \\ &\quad + 18A(R)C(R)Rr_2^3 + 9A(R)R^2r_2^3 - 9C(R)R^2r_2^3 - 3R^3r_2^3 \end{aligned} \quad (42)$$

$$F_3 = -3A(R)C(R)R^3r_2^3 \quad (43)$$

$$\begin{aligned} G_1 &= -18A(R)C(R)r_2^2 - 9A(R)Rr_2^2 + 18C(R)Rr_2^2 + 9R^2r_2^2 \\ &\quad - 18C(R)r_2^3 - 9Rr_2^3 \end{aligned} \quad (44)$$

$$\begin{aligned} G_2 &= 9A(R)C(R)R^2r_2^2 + 3A(R)R^3r_2^2 - 3C(R)R^3r_2^2 - 18A(R)C(R)Rr_2^3 \\ &\quad - 9A(R)R^2r_2^3 + 9C(R)R^2r_2^3 + 3R^3r_2^3 + 6A(R)C(R)r_2^4 \\ &\quad + 3A(R)Rr_2^4 - 6C(R)Rr_2^4 - 3R^2r_2^4 \end{aligned} \quad (45)$$

$$G_3 = +3A(R)C(R)R^3r_2^3 - 3A(R)C(R)R^2r_2^4 - A(R)R^3r_2^4 + C(R)R^3r_2^4. \quad (46)$$

We get the following equation

$$F_0 + F_1K_2^2 + F_2K_2^4 + F_3K_2^6 + G_1K_1^2 + G_2K_1^2K_2^2 + G_3K_2^4K_1^2 = 0. \quad (47)$$

It gives  $K_1^2$  in terms of  $A(R), B(R), C(R)$  and  $K_2^2, r_2$ . Then  $r_1$  is determined ( $0 < r_1 < r_2$ ) through Eq. (27). The value of  $K_0$  is such that  $K_0r_1 = x_{1,n_1}$ , whatever  $n_1$ , forcing  $r_1$  to be a zero of  $\psi_1$ .

For scattering data corresponding to a potential in the considered class, namely constant on intervals determined from zeros of the regular solution, the phase-equivalent potentials all correspond to a permutation of the zeros of the regular solution with the constraint that  $(\forall \ell = 1, \dots, N-1) \quad r_{\ell, n_\ell} < r_{\ell+1, n_{\ell+1}}$  and  $r_{N, n_N} < R$ .

The transparent potentials are zero on  $[r_1, R[$ . The values  $r_1$  and  $r_2$  are respectively zeros of the spherical Bessel functions  $j_1$  and  $j_2$ . The potentials are non-zero only on  $[0, r_1[$ . The value of  $K_0$ , such that  $K_0r_1 = x_1$  differs from  $k = \sqrt{E}$  if  $r_1 \neq x_1/k$ . This is what leads to the discrete ambiguities.

### Numerical Application

We construct a reference potential, which is constant on each interval  $[0, r_1[, [r_1, r_2[, [r_2, R[$ . Here, we choose  $v_0 = -3, K_0 = 2, v_1 = -8, K_1 = 3, v_2 = -4, K_2 = \sqrt{5}$ , together with  $r_1 = 2.24670598$  and  $r_2 = 2.6958027$ , which reduce  $\psi_1$  and  $\psi_2$  to zero, respectively. The phase-shifts  $\delta_j, j = 0, 1, 2$  of this potential are calculated at  $E = 1$  by means of  $A(R), B(R), C(R)$ .

Phase equivalent potentials are found by fitting piecewise constant potentials to these phase-shifts or more explicitly  $A(R), B(R), C(R)$ . First, the values of  $K_2^2$  and  $v_2$  are equal to the starting values  $v_2 = -4, K_2 = \sqrt{5}$  according to Eq. (34) or Eq. (35). Then,  $r_2$  is determined from Eqs. (32,33) with the constraint that  $r_2 < R$ . The value of  $K_1$  is obtained from Eq. (39), and  $r_1$  is determined ( $0 < r_1 < r_2$ ) through Eq. (27). The value of  $K_0$  is such that  $K_0r_1 = x_{1,n_1}$ ,  $r_1$  being a zero of  $\psi_1$ .

Once the phase equivalent potential is determined, the corresponding Schrödinger equation is solved for each partial wave. It yields the respective positions of the zeros of the regular solution. The resolution of Eq. (7) yields a phase-shift from which one subtracts the phase-shift of the reference potential.

A sample of solutions for  $R = 5.5$  and  $R = 10.0$  are displayed in Tables 3 and 4, respectively. The results underline a classification according to three indices :

- $n$ , the difference between the phase calculated directly from Eq. (7) and the exact value divided by  $\pi$ . It is found to be the same for the three waves  $\ell = 0, 1, 2$ .
- $n_1$ , numbering the zero of regular solution of the  $\ell = 1$  wave,
- $n_2$ , numbering the zero of the regular solution of the  $\ell = 2$  wave.

The reference potential is recovered for the  $\{n, n_1, n_2\} = \{0, 1, 1\}$  case. Moreover, in each subclass of fixed  $n$ , the solution can be ordered according to  $\{n_1, n_2\}$  in such a way that  $n_2 \geq n_1$ .

$n$	$n_1$	$n_2$	$K_0$	$r_1$	$K_1$	$r_2$
-1	1	1	1.1802	3.8073	3.9070	4.1795
0	1	1	2.0000	2.2467	3.0000	2.6958
0	1	2	1.4985	2.9986	3.9070	4.1795
0	2	2	2.0291	3.8073	3.9070	4.1795
1	1	1	8.9280	0.5033	2.6010	0.8710
1	1	2	3.8906	1.1549	3.0000	2.6958
1	1	3	2.0551	2.1865	3.9070	4.1795
1	2	2	3.4385	2.2467	3.0000	2.6958
1	2	3	2.5763	2.9986	3.9070	4.1795
1	3	3	2.8640	3.8073	3.9070	4.1795
2	1	4	3.2924	1.3648	3.9070	4.1795
2	2	2	15.3494	0.5033	2.6010	0.8710
2	2	3	6.6888	1.1549	3.0000	2.6958
2	2	4	3.5332	2.1865	3.9070	4.1795
2	3	3	4.8534	2.2467	3.0000	2.6958
2	3	4	3.6364	2.9986	3.9070	4.1795
2	4	4	3.6945	3.8073	3.9070	4.1795
3	1	5	9.2930	0.4835	3.9070	4.1795
3	2	5	5.6605	1.3648	3.9070	4.1795
3	3	3	21.6656	0.5033	2.6010	0.8710
3	3	4	9.4412	1.1549	3.0000	2.6958
3	3	5	4.9870	2.1865	3.9070	4.1795
3	4	4	6.2608	2.2467	3.0000	2.6958
3	4	5	4.6909	2.9986	3.9070	4.1795
3	5	5	4.5231	3.8073	3.9070	4.1795
4	2	6	15.9769	0.4835	3.9070	4.1795
4	3	6	7.9897	1.3648	3.9070	4.1795
4	4	4	27.9484	0.5033	2.6010	0.8710
4	4	5	12.1791	1.1549	3.0000	2.6958
4	4	6	6.4332	2.1865	3.9070	4.1795
4	5	5	7.6649	2.2467	3.0000	2.6958
4	5	6	5.7429	2.9986	3.9070	4.1795
4	6	6	5.3506	3.8073	3.9070	4.1795

Table 3: Phase-equivalent potential to the reference potential of line 2. The range parameter is  $R = 5.5$ .  $K_j^2 = k^2 - v_j$ ,  $k = 1$ . The classification is made according to indices  $n, n_1, n_2$  (see text)

$n$	$n_1$	$n_2$	$K_0$	$r_1$	$K_1$	$r_2$
-2	1	1	0.8310	5.4072	7.1067	5.6213
-1	1	1	1.1802	3.8073	3.9070	4.1795
-1	1	2	0.9051	4.9648	7.1067	5.6213
-1	2	2	1.4287	5.4072	7.1067	5.6213
0	1	1	2.0000	2.2467	3.0000	2.6958
0	1	2	1.4985	2.9986	3.9070	4.1795
0	1	3	0.9936	4.5223	7.1067	5.6213
0	2	2	2.0291	3.8073	3.9070	4.1795
0	2	3	1.5560	4.9648	7.1067	5.6213
0	3	3	2.0166	5.4072	7.1067	5.6213
1	1	1	8.9279	0.5033	2.6010	0.8710
1	1	2	3.8905	1.1550	3.0000	2.6958
1	1	3	2.0551	2.1865	3.9070	4.1795
1	1	4	1.1014	4.0798	7.1067	5.6213
1	2	2	3.4385	2.2467	3.0000	2.6958
1	2	3	2.5763	2.9986	3.9070	4.1795
1	2	4	1.7082	4.5223	7.1067	5.6213
1	3	3	2.8640	3.8073	3.9070	4.1795
1	3	4	2.1963	4.9648	7.1067	5.6213
1	4	4	2.6014	5.4072	7.1067	5.6213
2	1	4	3.2924	1.3648	3.9070	4.1795
2	1	5	1.2354	3.6371	7.1067	5.6213
2	2	2	15.3492	0.5033	2.6010	0.8710
2	2	3	6.6888	1.1550	3.0000	2.6958
2	2	4	3.5332	2.1865	3.9070	4.1795
2	2	5	1.8935	4.0798	7.1067	5.6213
2	3	3	4.8534	2.2467	3.0000	2.6958
2	3	4	3.6364	2.9986	3.9070	4.1795
2	3	5	2.4112	4.5223	7.1067	5.6213
2	4	4	3.6945	3.8073	3.9070	4.1795
2	4	5	2.8332	4.9648	7.1067	5.6213
2	5	5	3.1848	5.4072	7.1067	5.6213

Table 4: Same as Table 3 for a range parameter  $R = 10.0$ . The reference potential appears on line 5.

### 3.4 4 phase-shifts $\ell = 0, 1, 2, 3$

We begin the procedure by determining the value of  $K_3$  as a function of  $r_3$ . On the interval  $[r_3, R]$ , the regular solution  $\psi_3$ , which vanishes at  $r = r_3$ , is a linear combination of  $u_3(K_3 r)$  and  $w_3(K_3 r)$  :

$$\psi_3(k, r) = u_3(K_3 r)w_3(K_3 r_3) - u_3(K_3 r_3)w_3(K_3 r) . \quad (48)$$

According to the definition (6), the ratio  $(\psi_3/\psi'_3)(k, R) = D(R)$  at  $r = R$ . It yields

$$\begin{aligned} \frac{\tan(K_3(R - r_3))}{K_3} &= \frac{\mathcal{N}_3}{\mathcal{D}_3} & K_3 &= \sqrt{K_3^2} \\ \frac{\tanh(K_3^*(R - r_3))}{K_3^*} &= \frac{\mathcal{N}_3}{\mathcal{D}_3} & K_3^* &= \sqrt{-K_3^2} , \end{aligned} \quad (49)$$

where  $\mathcal{N}_3, \mathcal{D}_3$  are given in Appendix A. We first check if the potential is simply soluble. Looking at Eqs.(21,34), we expect

$$K_3^2 = \frac{9C(R)D(R) + 3C(R)R - 3D(R)R - R^2}{C(R)D(R)R^2} \quad (50)$$

or, equivalently, provided that  $R$  is not a zero of  $\psi_2$  or  $\psi_3$

$$\begin{aligned} K_3^2 &= k^2 + \frac{\mathcal{N}}{\mathcal{D}} \\ \mathcal{N} &= k^7 R^5 \sin(\delta_2 - \delta_3) \\ \mathcal{D} &= ((3 - k^2 R^2) \sin(kR + \delta_2) - 3kR \cos(kR + \delta_2)) \\ &\quad \times (3(5 - 2k^2 R^2) \sin(kR + \delta_3) + kR(k^2 R^2 - 15) \cos(kR + \delta_3)) . \end{aligned} \quad (51)$$

The equation (50) is verified if and only if  $(\psi_2/\psi'_2)(r = r_3) = C(r_3) = r_3/3$ . This can be proved in the following way. On the interval  $[r_3, R]$ , we express  $\psi_2$  as

$$\psi_2(k, r) = \alpha(K_3)u_2(K_3 r) + \beta(K_3)w_2(K_3 r) . \quad (52)$$

The ratio  $\alpha(K_3)/\beta(K_3)$  is determined in terms of  $C(r_3)$  and  $C(R)$ , which yields

$$\begin{aligned} \frac{\tan(K_3(R - r_3))}{K_3} &= \frac{\mathcal{N}'_3}{\mathcal{D}'_3} & K_3 &= \sqrt{K_3^2} \\ \frac{\tanh(K_3^*(R - r_3))}{K_3^*} &= \frac{\mathcal{N}'_3}{\mathcal{D}'_3} & K_3^* &= \sqrt{-K_3^2} , \end{aligned} \quad (53)$$

where  $\mathcal{N}'_3$  and  $\mathcal{D}'_3$  are given in Appendix A.

By equating  $\mathcal{N}_3/\mathcal{D}_3 = \mathcal{N}'_3/\mathcal{D}'_3$ , and taking  $K_3^2$  from (50), we find indeed  $C(r_3) = r_3/3$ .

It remains to actually calculate  $C(r_3)$ . To do so, we first note that  $D(r_1) = 5r_1/(x_{1,n_1}^2 - 15)$ . Here, use is made of the fact that  $K_0 r_1 = x_{1,n_1}$  is a zero of  $\psi_1$  and thus  $\sin(K_0 r_1) = K_0 r_1 \cos(K_0 r_1)$ .

On the interval  $[r_1, r_2]$ ,  $\psi_3$  is written as

$$\psi_3(k, r) = \alpha(K_1)u_3(K_1 r) + \beta(K_1)w_3(K_1 r) . \quad (54)$$

The ratio  $\alpha(K_1)/\beta(K_1)$  is given as a function of  $D(r_1)$ , thus as a function of  $r_1, x_{1,n_1}$ . This ratio being known,  $D(r_2)$  can be calculated.

Taking into account Eq.(27),  $D(r_2)$  is rational fraction in terms of  $x_{1,n_1}, K_1^2, r_1, r_2$ .

Finally, on  $[r_2, r_3[$ ,  $\psi_3$  is given by

$$\psi_3(k, r) = \alpha(K_2)u_3(K_2r) + \beta(K_2)w_3(K_2r) . \quad (55)$$

Similarly to the preceding step, the ratio  $\alpha(K_2)/\beta(K_2)$  is a function of  $D(r_2)$ . Inserting its value in (55), and by using Eqs. (25) and (26) at  $R = r_3$ , we obtain a compact polynomial expression involving  $x_{1,n_1}, K_1^2, K_2^2, r_1, r_2, r_3, C(r_3)$ . By remembering that  $D(r_3) = 0$  and setting  $C(r_3) = r_3/3$ , the condition for a simple soluble model reads

$$(9 + 3K_1^2r_2^2 + K_1^4r_2^4)(225 + 45K_2^2r_2^2 + 6K_2^4r_2^4 + K_2^6r_2^6)(K_1^2r_1^2 - x_{1,n_1}^2) = 0 . \quad (56)$$

Setting  $K_1^2r_2^2 = 3x$ ,  $K_2^2r_2^2 = y$  and  $K_1r_1 = z$ , the polynomial in  $x$  has no real root, the polynomial in  $y$  has a single real root, namely  $K_2^2r_2^2 = -5.39254$ . The polynomial in  $z$  has root  $K_1r_1 = x_{1,n_1}$  ( $K_1^2 \geq 0$ ) implying  $K_0 = K_1$ . Consequently,  $C(r_3) = r_3/3$  occurs only in exceptional situations, and the model is not simply soluble.

Since the model is not simply soluble we let  $r_3$  run from 0 to  $R$ . Use is made of Eq. (49) and the first equation of Appendix A to determine  $K_3$ . The value  $K_3(R - r_3)$  is determined in intervals  $]m_3\pi, (m_3+1)\pi[$ , where  $m_3$  ( $m_3 \geq 0$ ) is fixed at each run. We have ensured that a single solution exists in each interval, with  $K_3$  being a continuous function of  $r_3$  and  $\psi_3(r_3) = 0$ . Then  $A(r_3), B(r_3), C(r_3)$  (see Eq. (6)) for  $\ell = 0, 1, 2$ , respectively, are determined from  $A(R), B(R), C(R)$  by using the fact that on  $[r_3, R[$  we have

$$\psi_\ell(k, r) = \alpha_\ell u_\ell(K_3r) + \beta_\ell w_\ell(K_3r) . \quad (57)$$

By comparison with the previous case of 3 phase shifts, for which  $K_1, K_2$  are independent on  $r_1, r_2$  respectively, many more solutions have to be considered.

We first look at transparent potentials, for which the exact phase is  $n\pi$  for all waves  $\ell = 0, 1, 2, 3$ .  
<sup>1</sup> The results are displayed in Tables 5 and 6 for  $R = 12$  and  $R = 15$ , respectively.

We denote by  $n$  the class of phase-equivalent potential, for which the exact phase is  $n\pi$  for all waves  $\ell = 0, 1, 2, 3$ . The present study emphasizes these classes. Each class is characterized by the set of numbers  $\{n_1, n_2, n_3\}$  where  $n_\ell$  is  $n_\ell$ -th zero of the regular solution of the Schrödinger equation for the  $\ell$ -wave. The configurations can be ordered according to  $\{n_1, n_2, n_3\}$  with  $n_1 \leq n_2 \leq n_3 \leq n_{max}$ . The number of configurations and  $n_{max}$  for fixed  $n$  increase with  $R$ .

Obviously, except for the lowest exact phase, different sets of indices  $n_1 \leq n_2 \leq n_3$  exist, with different  $K_j, v_j$  giving the same phase  $n\pi$ . Consequently, the exact phase is not a sufficient criterion to remove the phase-ambiguities, as suggested in ref.[5]. The size of the set associated to the index  $n$  increases with the value of  $n$ .

Obviously, for each  $n$ , the set of minimal size corresponds to  $\{n_1, n_2, n_3\} = \{1, 1, 1\}$ ; ( $n_{max} = 1$ ). For the lowest  $n$ , this set is the only solution. Note that in Table 5 and 6 for  $n = 0$ , all the solutions reduce to the same zero potential.

As a more realistic example, the reference potential is chosen as the starting potential with parameters  $E = k^2 = 1$ ,  $R = 10$ ,

$$\begin{aligned} \sqrt{k^2 - v_0} = K_0 &= 2 & 0 \leq r < r_1 = 2.24668994 \\ \sqrt{k^2 - v_1} = K_1 &= 3 & r_1 \leq r < r_2 = 2.699578766 \\ \sqrt{k^2 - v_2} = K_2 &= \sqrt{5} & r_2 \leq r < r_3 = 7.38655698 \\ \sqrt{k^2 - v_3} = K_3 &= \sqrt{2} & r_3 \leq r < 10 . \end{aligned} \quad (58)$$

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<sup>1</sup>Note that here we may encounter transparent potentials for which the fourth exact phases are  $n(\ell)\pi$ , with  $n(\ell)$  not the same for all the waves.

$n$	$n_1$	$n_2$	$n_3$	$K_0$	$r_1$	$K_1$	$r_2$	$K_2$	$r_3$	$K_3$	$R$
-1	1	1	1	0.808	5.561	0.736	7.259	0.607	9.703	0.714	12.000
0	1	1	1	1.000	4.493	1.000	5.763	1.000	6.988	1.000	12.000
0	1	1	2	1.000	4.493	1.000	5.763	1.000	10.417	1.000	12.000
0	1	2	2	1.000	4.493	1.000	9.095	1.000	10.417	1.000	12.000
0	2	2	2	1.000	7.725	1.000	9.095	1.000	10.417	1.000	12.000
1	1	1	1	1.141	3.936	1.249	4.968	1.483	5.716	1.310	12.000
1	1	1	2	1.171	3.838	1.251	4.864	1.398	8.081	1.240	12.000
1	1	1	3	1.174	3.827	1.237	4.863	1.340	10.681	1.191	12.000
1	1	2	2	1.155	3.892	1.264	7.530	1.538	8.316	1.313	12.000
1	1	2	3	1.161	3.870	1.240	7.579	1.423	10.728	1.233	12.000
1	1	3	3	1.160	3.872	1.272	10.021	1.599	10.812	1.318	12.000
1	2	2	2	1.119	6.901	1.355	7.933	1.765	8.616	1.421	12.000
1	2	2	3	1.135	6.805	1.292	7.881	1.537	10.784	1.288	12.000
1	2	3	3	1.126	6.863	1.367	10.231	1.853	10.907	1.430	12.000
1	3	3	3	1.079	10.102	1.878	10.889	3.562	11.232	2.029	12.000

Table 5: Transparent potentials such that the first four phase-shift are all equal to  $n\pi$ .  $K_j^2 = k^2 - v_j$ ;  $k = 1$ . The range of the potential  $R = 12$ . The index  $n_\ell$  is such that  $r_\ell$  is the  $n_\ell$ -th zero of the regular solution in the  $\ell$ -wave.

The functions  $A(R), B(R), C(R), D(R)$  are calculated, and the procedure developed in subsection 3.4 is used to determine phase-equivalent potentials.

The classification of the solutions is performed very much in the same way as for transparent potentials. Here  $n$  is the difference between the exact phases of the reference and the phase-equivalent potentials divided by  $\pi$ . A priori,  $n$  depends on  $\ell$ . As an example, the following phase-equivalent potential  $v_j = k^2 - K_j^2$

$$\begin{aligned}
K_0, r_1 &= 3.12729631, 1.43683507 \\
K_1, r_2 &= 1.39601562, 4.82494432 \\
K_2, r_3 &= 2.54346784, 4.97266696 \\
K_3, R &= 0.820438746, 10
\end{aligned} \tag{59}$$

is such that the first 3 values of  $n(\ell = 0) = n(\ell = 1) = n(\ell = 2) = -2$ , while  $n(\ell = 3) = -3$ . In this case, the 3 indices  $n_\ell$  are not ordered according to  $n_1 \leq n_2 \leq n_3$ . Actually,  $n_1 = 1, n_2 = 2, n_3 = 1$ .

Let us concentrate on potentials for which  $n$  is independent of  $\ell$ . Results are displayed in Table 7. The same conclusions apply to this ensemble as to the case of transparent potentials. We note that one of the  $n = 0$  solutions reproduces the reference potential.

The configuration  $n = 1$  for non transparent potentials is reported in Appendix B.

## 4 Generalization by means of the JWKB approximation

The method developed in the preceding sections is not suitable for an extension to a larger number of phase shifts, as it would require a tremendous numerical effort. In order to work beyond 5 phase-shifts and obtain some general results, use can be made of the JWKB approximation. We recall that in this



$n$	$n_1$	$n_2$	$n_3$	$K_0$	$r_1$	$K_1$	$r_2$	$K_2$	$r_3$	$K_3$	$R$
-2	1	1	1	0.641	7.008	0.538	9.297	0.456	13.285	0.705	15.000
-1	1	1	1	0.833	5.396	0.756	7.048	0.683	9.210	0.803	15.000
-1	1	1	2	0.835	5.384	0.781	6.992	0.740	13.542	0.869	15.000
-1	1	2	2	0.833	5.391	0.772	11.360	0.699	13.512	0.847	15.000
-1	2	2	2	0.847	9.117	0.713	10.998	0.632	13.458	0.810	15.000
0	1	1	1	1.000	4.493	1.000	5.763	1.000	6.988	1.000	15.000
0	1	1	2	1.000	4.493	1.000	5.763	1.000	10.417	1.000	15.000
0	1	1	3	1.000	4.493	1.000	5.763	1.000	13.698	1.000	15.000
0	1	2	2	1.000	4.493	1.000	9.095	1.000	10.417	1.000	15.000
0	1	2	3	1.000	4.493	1.000	9.095	1.000	13.698	1.000	15.000
0	1	3	3	1.000	4.493	1.000	12.323	1.000	13.698	1.000	15.000
0	2	2	2	1.000	7.725	1.000	9.095	1.000	10.417	1.000	15.000
0	2	2	3	1.000	7.725	1.000	9.095	1.000	13.698	1.000	15.000
0	2	3	3	1.000	7.725	1.000	12.323	1.000	13.698	1.000	15.000
0	3	3	3	1.000	10.904	1.000	12.323	1.000	13.698	1.000	15.000
1	1	1	1	1.120	4.013	1.234	5.058	1.395	5.831	1.229	15.000
1	1	1	2	1.155	3.892	1.240	4.928	1.328	8.307	1.176	15.000
1	1	1	3	1.160	3.873	1.227	4.917	1.284	10.987	1.139	15.000
1	1	1	4	1.154	3.892	1.207	4.952	1.247	13.807	1.111	15.000
1	1	2	2	1.142	3.935	1.244	7.630	1.393	8.485	1.206	15.000
1	1	2	3	1.151	3.902	1.227	7.651	1.320	11.044	1.155	15.000
1	1	2	4	1.148	3.915	1.205	7.731	1.269	13.815	1.121	15.000
1	1	3	3	1.149	3.912	1.239	10.227	1.382	11.136	1.183	15.000
1	1	3	4	1.145	3.924	1.211	10.386	1.306	13.829	1.136	15.000
1	1	4	4	1.148	3.915	1.226	12.896	1.366	13.850	1.161	15.000
1	2	2	2	1.109	6.967	1.317	8.027	1.524	8.803	1.264	15.000
1	2	2	3	1.127	6.857	1.272	7.948	1.392	11.150	1.187	15.000
1	2	2	4	1.128	6.846	1.235	7.966	1.313	13.832	1.139	15.000
1	2	3	3	1.119	6.902	1.299	10.446	1.491	11.281	1.229	15.000
1	2	3	4	1.123	6.879	1.247	10.567	1.366	13.850	1.161	15.000
1	2	4	4	1.122	6.886	1.275	12.991	1.456	13.880	1.197	15.000
1	3	3	3	1.090	10.003	1.476	10.988	1.840	11.651	1.365	15.000
1	3	3	4	1.103	9.889	1.337	10.967	1.520	13.899	1.223	15.000
1	3	4	4	1.097	9.943	1.412	13.218	1.720	13.955	1.299	15.000
1	4	4	4	1.055	13.338	2.302	13.994	4.102	14.284	1.993	15.000

Table 6: Same as Table 5 but for  $R = 15$

$n$	$n_1$	$n_2$	$n_3$	$K_0$	$r_1$	$K_1$	$r_2$	$K_2$	$r_3$	$K_3$	$R$
-3	1	1	1	1.007	4.464	1.825	5.219	2.072	5.479	0.877	10.000
-2	1	1	1	1.267	3.546	2.222	4.164	2.308	4.391	1.294	10.000
-2	1	1	2	1.364	3.295	2.200	3.913	1.996	5.856	0.934	10.000
-2	1	2	2	1.339	3.355	2.174	5.469	2.004	5.771	0.920	10.000
-2	2	2	2	1.497	5.161	2.621	5.712	1.990	6.042	0.968	10.000
-1	1	1	1	1.464	3.069	2.697	3.594	2.541	3.782	2.847	10.000
-1	1	1	2	1.635	2.749	2.657	3.261	2.278	4.959	1.413	10.000
-1	1	1	3	1.771	2.538	2.640	3.048	2.044	6.597	1.099	10.000
-1	1	2	2	1.626	2.763	2.627	4.513	2.274	4.769	1.369	10.000
-1	1	2	3	1.781	2.523	2.626	4.273	1.999	6.235	1.008	10.000
-1	1	3	3	1.710	2.627	2.553	5.680	1.990	6.006	0.961	10.000
-1	2	2	2	1.756	4.400	3.146	4.860	2.290	5.130	1.457	10.000
-1	2	2	3	1.884	4.100	3.055	4.568	2.031	6.510	1.075	10.000
-1	2	3	3	1.800	4.291	2.955	5.853	1.996	6.193	0.999	10.000
-1	3	3	3	1.954	5.581	3.435	6.011	2.011	6.359	1.037	10.000
0	1	1	1	1.617	2.779	3.034	3.235	2.752	3.390	2.073	10.000
0	1	1	2	1.782	2.522	3.076	2.967	2.544	4.464	1.869	10.000
0	1	1	3	1.905	2.359	3.071	2.801	2.389	5.798	1.670	10.000
0	1	1	4	2.000	2.247	3.000	2.696	2.236	7.387	1.414	10.000
0	1	2	2	1.813	2.478	3.056	3.982	2.529	4.191	1.789	10.000
0	1	2	3	2.009	2.236	3.094	3.721	2.320	5.391	1.531	10.000
0	1	2	4	2.181	2.061	3.081	3.552	2.118	6.960	1.220	10.000
0	1	3	3	2.004	2.242	3.046	4.801	2.285	5.069	1.441	10.000
0	1	3	4	2.211	2.033	3.067	4.576	2.032	6.517	1.077	10.000
0	1	4	4	2.105	2.135	2.953	5.853	1.996	6.193	0.998	10.000
0	2	2	2	1.921	4.021	3.674	4.417	2.561	4.634	1.923	10.000
0	2	2	3	2.066	3.739	3.615	4.139	2.382	5.763	1.656	10.000
0	2	3	3	2.063	3.744	3.521	5.057	2.312	5.331	1.513	10.000
0	2	2	4	2.205	3.503	3.504	3.912	2.182	7.208	1.324	10.000
0	2	3	4	2.239	3.450	3.450	4.784	2.063	6.700	1.130	10.000
0	2	4	4	2.131	3.625	3.313	5.975	2.007	6.322	1.028	10.000
0	3	3	3	2.207	4.941	4.118	5.301	2.348	5.576	1.590	10.000
0	3	3	4	2.360	4.621	3.940	4.995	2.099	6.879	1.190	10.000
0	3	4	4	2.242	4.864	3.763	6.096	2.022	6.447	1.059	10.000
0	4	4	4	2.405	5.848	4.267	6.199	2.037	6.553	1.087	10.000

Table 7: Same as Tables 5 and 6 but for a non zero starting potential exhibited between the double lines.

case the phase-shift is given by:

$$\tilde{\delta}(\ell, k) = \lim_{r \rightarrow \infty} \left( \int_{rt(\ell, k)}^r K(\ell, k, r') dr' - \int_{\tilde{rt}(\ell, k)}^r \tilde{K}(\ell, k, r') dr' \right) \quad (60)$$

(remember that  $E = k^2$ ). For the type of potential considered in this work, we have

$$K(\ell, k, r) = \sqrt{k^2 - v(r) - \frac{\ell(\ell+1)}{r^2}} \quad \tilde{K}(\ell, k, r) = \sqrt{k^2 - \frac{\ell(\ell+1)}{r^2}}. \quad (61)$$

In (60),  $rt(\ell, k)$  is “the” largest turning point relative to the function  $K$  considered, while  $\tilde{rt}(\ell, k) = \sqrt{\ell(\ell+1)}/k$ .

The JWKB phase  $\tilde{\delta}(\ell, k)$  has been calculated for the transparent potentials of Table 5, and compared to the exact phase. In case of the null potential, the JWKB estimate is obviously exact. For non zero potentials, the relative differences

$$D_\ell = \left| \frac{\delta(\ell, k) - \tilde{\delta}(\ell, k)}{\delta(\ell, k)} \right| \quad (62)$$

are most of the time better than 1 %, but as large as 2 % in a few instances.

Furthermore, the JWKB method can be shown to predict the position of the zeros of the regular solution. To this aim, the regular solution is written

$$\psi_\ell(k, r) \propto \sin \left( \int_{rt(\ell, k)}^r \sqrt{k^2 - v(r') - \frac{\ell(\ell+1)}{r'^2}} dr' \right). \quad (63)$$

Clearly, for  $\hat{r}_n(\ell, k)$  to be the  $n$ th zero of the regular solution, the following condition has to be satisfied :

$$\int_{rt(\ell, k)}^{\hat{r}_n(\ell, k)} \sqrt{k^2 - v(r') - \frac{\ell(\ell+1)}{r'^2}} dr' = n\pi. \quad (64)$$

Actually, the quantity to be considered is

$$I_\ell = \frac{1}{\pi} \left( \int_{rt(\ell, k)}^{\hat{r}_n(\ell, k)} \sqrt{k^2 - v(r') - \frac{\ell(\ell+1)}{r'^2}} dr' \right) + c_\ell. \quad (65)$$

The levelling value  $c_\ell$  is given by

$$c_\ell = 1 - \frac{1}{\pi} \int_{\sqrt{\ell(\ell+1)}/k}^{r_1(\ell)/k} \sqrt{k^2 - \frac{\ell(\ell+1)}{r'^2}} dr', \quad (66)$$

where  $r_1(\ell)$  is the first zero of the spherical Bessel  $j_\ell(z)$ . The quantity  $c_\ell$  does not depend on the value of  $k$ . It is designed in a way that here  $I_\ell$  is an integer and yields the exact result 1 for the first zero in the absence of potential.

Calculations have been performed for the transparent potentials of Table 5. The ratio between the exact values and the JWKB estimates show an excellent agreement. The deviation with respect to unity is most of the time better than 1 %, with few exceptions of the order of a few %. In this case,  $I_\ell$  is close to an integer and represents the position of the  $n$ th zero of the regular solution for the  $\ell$  wave.

Comforted by the good quantitative agreement of the above calculations, the JWKB approximation can be used to ask questions on a general basis. Note that  $N \geq 3$  is required, otherwise the model is

simple soluble and the JWKB approximation is not necessary. For instance, it is possible to predict the lowest value of  $n$  for a set of  $N + 1$  phase-shifts, and a fixed potential range  $R$ . According to the results displayed in the previous Tables, it occurs for the configuration in which all the  $n_N = 1$ . Thus, it is sufficient to consider the largest value of  $N$ .

The data are the number of phase shifts  $N + 1$  the energy  $E = k^2$  and the potential range  $R$ . Recalling the previous conventions,  $\hat{r}_1(N, k)$  denotes the first zero of the regular solution for the  $N$ -wave, supposed to be strictly lower than  $R$ .

We first consider transparent potentials. By definition, we have

$$n = \frac{Q_1 + Q_2 - Q_3}{\pi}, \quad (67)$$

with

$$Q_1 = \int_{rt(N, k)}^{\hat{r}_1(N, k)} \sqrt{k^2 - v(r') - \frac{N(N+1)}{r'^2}} dr', \quad (68)$$

$$Q_2 = \int_{\hat{r}_1(N, k)}^R \sqrt{k^2 - v(r') - \frac{N(N+1)}{r'^2}} dr', \quad (69)$$

and

$$Q_3 = \int_{\sqrt{N(N+1)/k}}^R \sqrt{k^2 - \frac{N(N+1)}{r'^2}} dr'. \quad (70)$$

The first contribution yields

$$Q_1 = \pi - c_N \pi. \quad (71)$$

However,  $c_N$  is defined by Eq.(66), which yields

$$Q_1 = \int_{\sqrt{N(N+1)/k}}^{r_1(N)/k} \sqrt{k^2 - \frac{N(N+1)}{r'^2}} dr'. \quad (72)$$

The potential is constant on the interval  $[\hat{r}_1(N, k), R]$  so that the two next integrals are analytical and read

$$\begin{aligned} Q_2 &= \sqrt{K_N^2 R^2 - N(N+1)} - \sqrt{N(N+1)} \arctan \sqrt{\frac{K_N^2 R^2 - N(N+1)}{N(N+1)}} \\ &- \sqrt{K_N^2 \hat{r}_1^2(N, k) - N(N+1)} - \sqrt{N(N+1)} \arctan \sqrt{\frac{K_N^2 \hat{r}_1^2(N, k) - N(N+1)}{N(N+1)}} \end{aligned} \quad (73)$$

and

$$Q_3 = \sqrt{k^2 R^2 - N(N+1)} - \sqrt{N(N+1)} \arctan \sqrt{\frac{k^2 R^2 - N(N+1)}{N(N+1)}}. \quad (74)$$

It is clear that for the configurations having all  $n_N = 1$ , the value of  $n$  is ordered by the value of  $\hat{r}_1(N, k)$ . The lowest configuration corresponds to  $\hat{r}_1(N, k)$  being the closest to  $R$ . The absolute limit is given, provided  $kR$  high enough and  $R \neq \hat{r}_1(N, k)$ , by

$$n_{\min} > \frac{Q_1 - Q_3}{\pi}, \quad (75)$$

because  $Q_2$  is strictly positive. In a compact form taking into account the equation (72) we obtain

$$n_{\min} > -\frac{1}{\pi} \int_{r_1(N)/k}^R \sqrt{k^2 - \frac{N(N+1)}{r'^2}} dr'. \quad (76)$$

Actually,  $n_{\min}$  is integer and we have to take

$$n_{\min} \geq 1 + \left[ -\frac{1}{\pi} \int_{r_1(N)/k}^R \sqrt{k^2 - \frac{N(N+1)}{r'^2}} dr' \right]. \quad (77)$$

Here,  $[x]$  denotes the integer part of  $x$ , i.e. the integer  $m_x$  such that  $m_x \leq x < m_x + 1$ . In the present case we use the property that any integer  $m$  which satisfies  $m > x$  satisfies  $m \geq [x] + 1$ .

If we take as an example the upper case of Tables 5 and 6, the numerical estimate of Eq.(76) yields  $n_{\min} > -1.47631$  and  $n_{\min} > -2.39885$  for  $R = 12$  and  $R = 15$ , respectively. The integer values satisfy  $n_{\min} \geq -1$  and  $n_{\min} \geq -2$ , respectively, in agreement with the values of Tables 5 and 6 for  $R = 12$  and  $R = 15$ .

On the other hand, by looking at the Tables 5,6 and 7, the maximum value of  $n_3$  depends on  $n$ . It obeys the following recurrence relation

$$n_{3,\max}(n+1) = n_{3,\max}(n) + 1. \quad (78)$$

For the transparent potentials,  $n_{3,\max}(0)$  is given by the number of zeros of the regular spherical function  $j_3(kr)$  below  $kR$  (we exclude the exceptional situation where a zero of  $j_3$  is equal to  $kR$ ). For instance let us consider  $R = 12, k = 1$ . There are two zeros of  $j_3$  below 12, namely 6.988 and 10.417. They can be found in the Table 5 concerning the configuration  $n = 0$  (zero potential). This implies that  $n_{3,\max}(0) = 2$  so that  $n_{3,\max}(-1) = n_{3,\max}(0) - 1 = -1$  (see Table 5).

A similar example is found from Table 6. Three zeros of  $j_3$  are found below  $kR = 15$  : 6.988, 10.417 and 13.698. It yields  $n_{3,\max}(0) = 3$ , which implies that  $n_{3,\max}(-2) = 1$ . We recover the lowest configuration of the Table 6.

Yet a further example is provided us by  $R = 21, k = 1$ . Five zeros of  $j_3$  exist below 21 : 6.988, 10.417, 13.698, 16.924 and 20.122. It gives  $n_{3,\max}(0) = 5$  and  $n_{3,\max}(-4) = 1$ . This configuration has been confirmed numerically. It corresponds to the potential ( $v_j = k^2 - K_j^2 = 1 - K_j^2$ ):

$$\begin{aligned} K_0, r_1 &= 0.436410526, 10.2962892 \\ K_1, r_2 &= 0.34393, 13.836594 \\ K_2, r_3 &= 0.321277, 20.0371883 \\ K_3, R &= 0.791895048, 21. \end{aligned} \quad (79)$$

The exact phases of this potential are  $-4\pi$  for  $\ell = 0 - 3$ .

Taking the same case,  $R = 21, k = 1$ , but for 5 phases instead of 4, 4 zeros of  $j_4$  are situated below 21, namely 8.183, 11.705, 15.040 and 18.301. As expected,  $n_{4,\max}(0) = 4$  and  $n_{4,\max}(-3) = 1$ . The lowest configuration is given by the following potential ( $v_j = k^2 - K_j^2 = 1 - K_j^2$ ):

$$\begin{aligned} K_0, r_1 &= 0.536260007, 8.37916127 \\ K_1, r_2 &= 0.218548173, 11.03235 \\ K_2, r_3 &= 0.360733527, 13.5849 \\ K_3, r_4 &= 0.793613806, 14.7336 \\ K_4, R &= 0.266900301, 21. \end{aligned} \quad (80)$$

Its exact phases are  $-3\pi$  for  $\ell = 0 - 4$

Beyond 5 phases, finding numerical solutions becomes too tedious to be interesting.

Taking into account the results for 4 and 5 phases ( $\ell = 0, 1, 2, 3$  and  $\ell = 0, 1, 2, 3, 4$ ), we propose the following conjecture :

Consider models which are not simply soluble, i.e. number of phases  $N + 1$  strictly greater than 3, and configurations such that  $n$  is the same for all partial waves, the minimal value of  $n$  is given by  $n_{\min} = -n(j_N) + 1$ , where  $n(j_N)$  denotes the number of zeros of the regular spherical Bessel function  $j_N$  strictly smaller than  $kR$ ,  $kR$  not being a zero of  $j_N$  and high enough to ensure  $n(j_N) \neq 0$ .

As stated in the preceding section, our attention has been focused on cases such that the phase difference is  $n\pi$ , independently of  $\ell$ . However, phase equivalent potentials with  $n$  depending on  $\ell$  can be constructed. Two examples are quoted below for transparent potentials.

The first example corresponds to 4 phase shifts, namely  $\delta_0 = \delta_1 = \delta_2 = 2\pi$  and  $\delta_3 = \pi$ . The phase equivalent potential reads

$$\begin{aligned} K_0, r_1 &= 6.80886924, 0.659934689 \\ K_1, r_2 &= 1.2151, 12.8731444 \\ K_2, r_3 &= 1.34685, 13.8434964 \\ K_3, R &= 1.15320976, 15. \end{aligned} \tag{81}$$

The values of  $n_\ell$  are and  $n_1 = 1, n_2 = 5, n_3 = 4$ .

The second case includes 5 phase shifts, namely  $\delta_0 = \delta_1 = \delta_2 = \pi$  and  $\delta_3 = \delta_4 = 0$ . The phase equivalent potential is given by

$$\begin{aligned} K_0, r_1 &= 7.43592354, 0.604283917 \\ K_1, r_2 &= 1.08584, 5.38409355 \\ K_2, r_3 &= 1.13832, 12.2708388 \\ K_3, r_4 &= 1.07046, 13.398 \\ K_4, R &= 0.814872, 21. \end{aligned} \tag{82}$$

The values of  $n_\ell$  are  $n_1 = 1, n_2 = 2, n_3 = n_4 = 3$ .

A similar bound is valid for non transparent potentials. We remind the reader that in this case  $n\pi$  is the difference between the exact phase of the equivalent potential and the exact phase of the reference potential. For non transparent potential the above inequality becomes

$$n_{\min} > \frac{1}{\pi} \left( \pi - c_N \pi - \int_{rt(N,k)}^R \sqrt{k^2 - v^{(s)}(r') - \frac{N(N+1)}{r'^2}} dr' \right). \tag{83}$$

Here,  $v^{(s)}$  is the starting reduced potential and  $rt(N, k)$  the corresponding largest turning point. On the basis of the results obtained for transparent potentials, we have

$$n_{\min} \geq 1 + \left[ -\frac{1}{\pi} \left( \int_{r_1(N)/k}^R \sqrt{k^2 - \frac{N(N+1)}{r'^2}} dr' + \delta(N, k) \right) \right]. \tag{84}$$

The quantity  $\delta(N, k)$  is either the exact phase of the starting potential calculated from Eq. (7) or the JWKB approximated phase Eq. (60).

For  $N = 3$  and the starting potential of Table 7 we obtain  $n_{\min} \geq 1 + [-4.08774] = -4$  and  $n_{\min} \geq 1 + [-4.01251] = -4$  according to the fact that the phase is exact or approximated. It has to be compared to the value  $n_{\min} = -3$  of Table 7.

Let us make a brief comment about the maximal number of nodes  $\tilde{N}$  of the wave functions (except the trivial zero at the origin) inside the potential. For a given  $\ell$ ,  $\tilde{N}$  is the number of zeros  $r_\ell$  of the regular solution of the Schrödinger equation below  $R$ . We have checked, from examples of Tables 5-7 and Appendix B, that for all potentials in the "class"  $n$  (such that  $(\delta(\ell, k) = n\pi, \ell = 0, 1, 2, 3)$ )  $\tilde{N}$  depends only on  $\ell, k, R, n$ . Moreover  $\tilde{N}(\ell, k, R, n)$  satisfies  $\tilde{N}(\ell, k, R, n+1) = \tilde{N}(\ell, k, R, n) + 1$ . In this sense,  $n$  is correlated to  $\tilde{N}$  as expected from the results of Sabatier [7], who argued that additional multiples of  $\pi$  were associated with additional nodes "inside" the potential. However the Sabatier's method, based on the Abel transform, cannot be directly applied here. Indeed the latter method requires the function  $r \mapsto r^2(k^2 - v(r))$  to be continuously differentiable [7, 8].

## 5 Conclusions

The present work is devoted to the study of discrete ambiguities. They arise when a specific parametrized expression is used as a potential, its parameters being fitted to a finite number of phase-shifts, or more directly to a scattering amplitude. To this aim, use is made of piecewise constant potentials, the intervals of which are defined by the zeros of the regular solution of the Schrödinger equation. These potentials generate a class of phase equivalent potentials with a phase ambiguity of  $n\pi$ ,  $n$  being an integer. The reference potential belongs obviously to this class.

A few examples have been investigated and solved exactly for 1-5 phase-shifts. The number of discrete ambiguities depends on the range of the potential. For a given value of  $n$ , the phase equivalent potentials can be ordered according to the position of the zeros of the regular solutions, except for the zero potential.

Note that the present study considers both positive and negative potentials. We have verified that in the case of attractive potentials having at least one bound state, the measurement of the ground state energy is a sufficient criterion to fix the potential and get a unique answer.

For a number of phase-shifts larger than 5, the numerical effort becomes rapidly intractable for an exact solution. In this case, use can be made of the JWKB approximation. This last allows to derive a lower bound for the minimal value  $n_{\min}$ .

Advantage has been taken at several places of the transparent potentials. They are defined as potentials having all their phase-shifts equal to  $n\pi$ . Particular attention has been given to the transparent potentials for which  $n$  is independent of  $\ell$ . Cases can occur, however, such that the exact phases are not the same for all waves. Then  $n$  depends on  $\ell$ .

Moreover, for transparent potentials, a conjecture has been proposed to determine  $n_{\min}$ .

Finally, we show that removing the phase-ambiguity as suggested by Drisko, Satchler and Bassel is not sufficient to determine the potential uniquely. Potentials reproducing the exact phases can differ by the numbering of the zeros of their regular solutions. Actually, the difference between the two approaches lies in the following. If a scattering amplitude is reproduced with a Woods-Saxon potential, for instance, the number of parameters is limited, and the large number of phase-shift to be fitted ensure the quasi unique determination of the parameters. Thus, phase ambiguities can be observed, and possibly removed. On the other hand, piecewise constant potentials generate a much large functional space. Consequently, ambiguities arise also from the choice of the intervals. This behavior has been emphasised by Lombard and Wilkin [21], in analysing high energy differential cross sections via the Glauber model. In this case, there is no phase ambiguity but the potential is defined as the statistical average over a large ensemble of different piecewise constant potentials fitting the data.

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## Appendix A

In this Appendix we report the expression of  $\mathcal{N}_3, \mathcal{D}_3$  of Eq.(49) and  $\mathcal{N}'_3, \mathcal{D}'_3$  of Eq.(53)

$$\begin{aligned}
\mathcal{N}_3 &= 675D(R)R + 225R^2 - 90D(R)K_3^2R^3 - 15K_3^2R^4 - 675D(R)r_3 \\
&- 225Rr_3 + 315D(R)K_3^2R^2r_3 + 90K_3^2R^3r_3 - 15D(R)K_3^4R^4r_3 \\
&- 270D(R)K_3^2Rr_3^2 - 90K_3^2R^2r_3^2 + 36D(R)K_3^4R^3r_3^2 \\
&+ 6K_3^4R^4r_3^2 + 45D(R)K_3^2r_3^3 + 15K_3^2Rr_3^3 \\
&- 21D(R)K_3^4R^2r_3^3 - 6K_3^4R^3r_3^3 + D(R)K_3^6R^4r_3^3 \\
\mathcal{D}_3 &= 675D(R) + 225R - 315D(R)K_3^2R^2 - 90K_3^2R^3 + 15D(R)K_3^4R^4 \\
&+ 675D(R)K_3^2Rr_3 + 225K_3^2R^2r_3 - 90D(R)K_3^4R^3r_3 - 15K_3^4R^4r_3 \\
&- 270D(R)K_3^2r_3^2 - 90K_3^2Rr_3^2 + 126D(R)K_3^4R^2r_3^2 \\
&+ 36K_3^4R^3r_3^2 - 6D(R)K_3^6R^4r_3^2 - 45D(R)K_3^4Rr_3^3 \\
&- 15K_3^4R^2r_3^3 + 6D(R)K_3^6R^3r_3^3 + K_3^6R^4r_3^3
\end{aligned} \tag{85}$$

$$\begin{aligned}
\mathcal{N}'_3 &= 9R^2r_3 - 9Rr_3^2 + 3K_3^2R^3r_3^2 - 3K_3^2R^2r_3^3 + 18Rr_3C(R) - 3K_3^2R^3r_3C(R) \\
&- 18r_3^2C(R) + 9K_3^2R^2r_3^2C(R) - 6K_3^2Rr_3^3C(R) + K_3^4R^3r_3^3C(R) + 18R^2C(r_3) \\
&- 18Rr_3C(r_3) + 6K_3^2R^3r_3C(r_3) - 9K_3^2R^2r_3^2C(r_3) + 3K_3^2Rr_3^3C(r_3) \\
&- K_3^4R^3r_3^3C(r_3) + 36RC(R)C(r_3) - 6K_3^2R^3C(R)C(r_3) - 36r_3C(R)C(r_3) \\
&+ 18K_3^2R^2r_3C(R)C(r_3) - 18K_3^2Rr_3^2C(R)C(r_3) + 3K_3^4R^3r_3^2C(R)C(r_3) \\
&+ 6K_3^2r_3^3C(R)C(r_3) - 3K_3^4R^2r_3^3C(R)C(r_3) \\
\mathcal{D}'_3 &= 9Rr_3 - 3K_3^2R^3r_3 + 9K_3^2R^2r_3^2 - 3K_3^2Rr_3^3 + K_3^4R^3r_3^3 + 18r_3C(R) \\
&- 9K_3^2R^2r_3C(R) + 18K_3^2Rr_3^2C(R) - 3K_3^4R^3r_3^2C(R) - 6K_3^2r_3^3C(R) \\
&+ 3K_3^4R^2r_3^3C(R) + 18RC(r_3) - 6K_3^2R^3C(r_3) + 18K_3^2R^2r_3C(r_3) - 9K_3^2Rr_3^2C(r_3) \\
&+ 3K_3^4R^3r_3^2C(r_3) - 3K_3^4R^2r_3^3C(r_3) + 36C(R)C(r_3) - 18K_3^2R^2C(R)C(r_3) \\
&+ 36K_3^2Rr_3C(R)C(r_3) - 6K_3^4R^3r_3C(R)C(r_3) - 18K_3^2r_3^2C(R)C(r_3) \\
&+ 9K_3^4R^2r_3^2C(R)C(r_3) - 6K_3^4Rr_3^3C(R)C(r_3) + K_3^6R^3r_3^3C(R)C(r_3)
\end{aligned} \tag{86}$$

## Appendix B

$n$	$n_1$	$n_2$	$n_3$	$K_0$	$r_1$	$K_1$	$r_2$	$K_2$	$r_3$	$K_3$	$R$
1	1	1	1	1.737	2.587	3.446	2.990	2.883	3.118	2.456	10.000
1	1	1	2	0.918	4.896	3.900	5.275	1.855	7.097	4.680	10.000
1	1	1	3	1.932	2.325	3.475	2.722	2.673	5.356	2.209	10.000
1	1	1	4	1.996	2.252	3.437	2.650	2.593	6.627	2.075	10.000
1	1	1	5	2.057	2.184	3.361	2.589	2.499	8.024	1.895	10.000
1	1	2	2	1.943	2.312	3.476	3.635	2.737	3.805	2.197	10.000
1	1	2	3	2.106	2.134	3.515	3.442	2.596	4.903	2.019	10.000
1	1	2	4	2.235	2.011	3.505	3.309	2.473	6.180	1.832	10.000
1	1	2	5	2.360	1.904	3.456	3.233	2.339	7.675	1.595	10.000
1	1	3	3	2.171	2.070	3.492	4.302	2.549	4.518	1.886	10.000
1	1	3	4	2.388	1.882	3.539	4.085	2.373	5.716	1.639	10.000
1	1	3	5	2.592	1.734	3.539	3.938	2.187	7.225	1.332	10.000
1	1	4	4	2.395	1.877	3.475	5.035	2.309	5.309	1.506	10.000
1	1	4	5	2.648	1.697	3.520	4.818	2.068	6.729	1.139	10.000
1	1	5	5	2.514	1.787	3.367	5.991	2.009	6.339	1.032	10.000
1	2	2	2	2.032	3.802	4.212	4.148	2.772	4.323	2.384	10.000
1	2	2	3	1.324	5.834	5.330	6.117	2.084	7.739	4.610	10.000
1	2	2	4	2.228	3.467	4.100	3.821	2.573	6.558	2.032	10.000
1	2	2	5	2.327	3.320	3.974	3.682	2.441	7.909	1.783	10.000
1	2	3	3	2.208	3.498	4.069	4.634	2.589	4.852	2.000	10.000
1	2	3	4	2.378	3.248	4.027	4.395	2.428	5.986	1.745	10.000
1	2	3	5	2.544	3.037	3.923	4.189	2.238	7.393	1.418	10.000
1	2	4	4	2.389	3.234	3.912	5.225	2.336	5.500	1.565	10.000
1	2	4	5	2.615	2.954	3.867	4.967	2.094	6.856	1.182	10.000
1	2	5	5	2.483	3.112	3.691	6.078	2.019	6.429	1.054	10.000
1	3	3	3	2.346	4.647	4.825	4.956	2.639	5.171	2.126	10.000
1	3	3	4	2.480	4.398	4.691	4.714	2.492	6.258	1.869	10.000
1	3	3	5	2.628	4.150	4.490	4.479	2.303	7.581	1.530	10.000
1	3	4	4	2.488	4.382	4.479	5.418	2.369	5.692	1.631	10.000
1	3	4	5	2.689	4.056	4.326	5.127	2.125	6.989	1.231	10.000
1	3	5	5	2.551	4.274	4.111	6.170	2.033	6.524	1.079	10.000
1	4	4	4	2.657	5.293	5.134	5.586	2.401	5.858	1.693	10.000
1	4	4	5	2.833	4.966	4.864	5.273	2.155	7.109	1.280	10.000
1	4	5	5	2.685	5.240	4.592	6.252	2.046	6.608	1.103	10.000
1	5	5	5	2.858	6.026	5.112	6.322	2.059	6.678	1.124	10.000

Table 8: This Table is an extension of Table 7 (see text).